The effect of the number of nodal diameters on non-linear interactions in two asymmetric vibration modes of a circular plate

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Abstract

In order to investigate the effect of the number of nodal diameters on non-linear interactions in asymmetric vibrations of a circular plate, a primary resonance of the plate is considered. The plate is assumed to have an internal resonance in which the ratio of the natural frequencies of two asymmetric modes is three to one. The response of the plate is expressed as an expansion in terms of the linear, free oscillation modes, and its amplitude is considered to be small but finite, and the method of multiple scales is used. In view of the corrected solvability conditions for the responses, it has been found that in order for the modes to interact, the ratio of the numbers of nodal diameters of two modes must be either three to one or one to one. In this study the one-to-one case, in which the modes have the same number of nodal diameters, is examined. The non-linear governing equations are reduced to a system of autonomous ordinary differential equations for amplitude and phase variables by means of the corrected solvability conditions. The steady state responses and their stability are determined by using this system. The result shows very complicated interactions between two modes by telling existence of non-vanishing amplitudes of the mode not directly excited.

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1. Introduction

Since Sridhar et al. [1] opened the doors to non-linear modal interactions of symmetric vibrations of circular plates, Hadian and Nayfeh [2], and Lee and Kim [3] followed their idea to investigate the interactions between symmetric modes of circular plates. Sridhar et al. [4] generalized the idea to derive solvability conditions that include asymmetric modes as well as

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symmetric modes. Yeo and Lee [5] found that these conditions were misderived, and then corrected the conditions. They observed that in the absence of internal resonance, the steady state response can have not only the form of standing wave but also the form of traveling wave. This observation is a remarkable contrast to Sridhar et al. [4], in which the steady state response can only have the form of standing wave. It is believed that the validity of this observation may be well supported by Nayfeh and Vakakis [6], who observed the coexistence of subharmonic standing and traveling waves in the case of subharmonic resonance.

Using the corrected solvability conditions, Lee and Yeo [7] investigated the interactions of circular plates on an elastic foundation with internal resonance ($\omega_{NM} \approx 3\omega_{CD}$), in which the first subscript refers to the number of nodal diameters and the second subscript the number of nodal circles including boundary. In view of the corrected solvability condition, it has been found that in order for two asymmetric modes to interact, the ratio of the numbers of nodal diameters of two modes must be either three to one ($N = 3C$) or one to one ($N = C$). They considered the case of $N = 3C$, which implies that the ratio of the numbers of nodal diameters of two modes is three to one.

In order to investigate the effect of the number of nodal diameters on non-linear interactions in asymmetric vibrations of the plate, in this study we examined the one-to-one case ($N = C$), in which the modes have the same number of nodal diameters. Dynamic analogue of von Karman equations is used to study a primary resonance of the plate. The response of the plate is expressed as an expansion in terms of the linear, free oscillation modes, and its amplitude is considered to be small but finite, and the method of multiple scales is used. The non-linear governing equations are reduced to a system of autonomous ordinary differential equations for amplitude and phase variables by means of the corrected solvability conditions. The steady state responses and their stability are determined by using this system.

2. Equations of motion and steady state responses

The equations governing the free, undamped oscillations of non-uniform circular plates were derived by Efstathiades [8]. These equations are simplified to fit the special case of uniform plates, and damping and forcing terms are added. Then the non-dimensionalized equations of motion of a circular plate on an elastic foundation shown in Fig. 1 can be given as follows [4,9]:

$$\frac{\partial^2 w}{\partial t^2} + (\nabla^4 + K)w = \varepsilon \left[L(w, F) - 2c \frac{\partial w}{\partial t} + p^*(r, \theta, t) \right], \tag{1}$$

$$\nabla^4 F = \left(\frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right)^2 - \frac{\partial^2 w}{\partial r^2} \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right), \tag{2}$$

where

$$\begin{aligned} L(w, F) = {} & \frac{\partial^2 w}{\partial r^2} \left(\frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} \right) + \frac{\partial^2 F}{\partial r^2} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \\ & - 2 \left(\frac{1}{r} \frac{\partial^2 F}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial F}{\partial \theta} \right) \left(\frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right), \end{aligned} \tag{3}$$
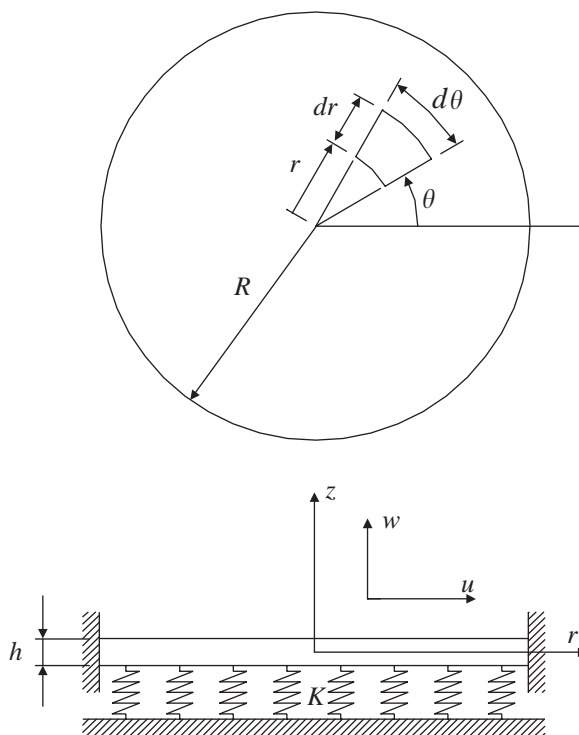
Fig. 1. Schematic diagram of a clamped circular plate on an elastic foundation.

$\varepsilon = 12(1 - v^2)h^2/R^2$, c is the damping coefficient, p^* is the forcing function, K is the stiffness of the foundation, v is the Poisson ratio, h is the thickness, R is the radius, w is the deflection of the middle surface, F is the force function which satisfies the in-plane equilibrium conditions (in-plane inertia is neglected), and

$$\nabla^4 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2. \tag{4}$$

The boundary conditions are developed for the plates which are clamped along a circular edge. For all t and θ,

$$w = 0, \quad \frac{\partial w}{\partial r} = 0, \quad \text{at } r = 1, \tag{5a, b}$$

$$\frac{\partial^2 F}{\partial r^2} - v\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} \right) = 0, \quad \text{at } r = 1, \tag{6a}$$

$$\frac{\partial^3 F}{\partial r^3} + \frac{1}{r}\frac{\partial^2 F}{\partial r^2} - \frac{1}{r^2}\frac{\partial F}{\partial r} + \frac{2+v}{r^2}\frac{\partial^3 F}{\partial r \partial \theta^2} - \frac{3+v}{r^3}\frac{\partial^2 F}{\partial \theta^2} = 0, \quad \text{at } r = 1, \tag{6b}$$

where R is assumed to be a non-dimensionalized radius, 1. In addition, it is necessary to require the solution to be bounded at $r = 0$.

The forcing function p^* is considered as follows:

$$p^*(r, \theta, t) = \left[\sum_{m=1}^{\infty} P_{0m}\phi_{0m} + 2 \sum_{n,m=1}^{\infty} P_{nm}\phi_{nm} \cos(n\theta + \tau_{nm}) \right] \cos \lambda t, \tag{7}$$

where the linear symmetric vibration modes $\phi_{nm}(r)$ correspond to the natural frequencies ω_{nm} (see Appendix A). In these expressions, the first subscript n refers to the number of nodal diameters and the second subscript m refers to the number of nodal circles including boundary. And λ is excitation frequency and P_{nm} are excitation amplitudes.

To obtain the first order approximate solution of the Eqs. (1)–(6) we use the method of multiple scales. We expand w and F as follows:

$$w(r, \theta, t; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j w_j(r, \theta, T_0, T_1, \ldots), \tag{8}$$

$$F(r, \theta, t; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j F_j(r, \theta, T_0, T_1, \ldots), \tag{9}$$

where $T_n = \varepsilon^n t$.

Following Sridhar et al. [4], we can have the first order solution as follows:

$$w_0 = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \phi_{nm}(r)u_{nm}(T_0, T_1, \ldots)e^{in\theta}, \quad u_{nm} = A_{nm}e^{i\omega_{nm}T_0} + \bar{B}_{nm}e^{-i\omega_{nm}T_0}, \tag{10}$$

where the ω_{nm} are linear natural frequencies, the $\phi_{nm}(r)$ are linear symmetric vibration modes (see Appendix A), and the responses A_{nm} and B_{nm} are complex functions of the all T_k for $k \geqslant 1$.

For a circular plate without an elastic foundation, i.e., the case of $K = 0$, Yeo and Lee [5] had corrected solvability conditions for the responses derived by Sridhar et al. [4]. Since value of K does not change solvability conditions at all, we refer to Yeo and Lee [5] for the conditions as follows:

$$-2i\omega_{kl}(D_1 A_{kl} + c_{kl}A_{kl}) + A_{kl}\left\{ \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{klnm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm}) - \gamma_{klkl}A_{kl}\bar{A}_{kl} \right\}$$

$$+ 2(1 - \delta_{k0})B_{kl}\left\{ \sum_{m=1}^{\infty} \hat{\gamma}_{klkm}A_{km}\bar{B}_{km} - \hat{\gamma}_{klkl}A_{kl}\bar{B}_{kl} \right\} + N_{kl}^A + R_{kl}^A = 0, \tag{11a}$$

$$2i\omega_{kl}(D_1 \bar{B}_{kl} + c_{kl}\bar{B}_{kl}) + \bar{B}_{kl}\left\{ \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{klnm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm}) - \gamma_{klkl}B_{kl}\bar{B}_{kl} \right\}$$

$$+ 2(1 - \delta_{k0})\bar{A}_{kl}\left\{ \sum_{m=1}^{\infty} \hat{\gamma}_{klkm}A_{km}\bar{B}_{km} - \hat{\gamma}_{klkl}A_{kl}\bar{B}_{kl} \right\} + N_{kl}^B + R_{kl}^B = 0, \tag{11b}$$

where $D_1 = \partial/\partial T_1$, δ_{k0} are Kronecker delta, c_{kl} are modal damping coefficients, $R_{kl}^{A,B}$ are terms due to internal resonances, if any, $N_{kl}^{A,B}$ are terms due to the external excitation, if any, and γ_{klnm} and $\hat{\gamma}_{klkm}$ are constants given in the Appendix A.

It is noted that these solvability conditions are different from those by Sridhar et al. [4]. Terms including expressions γ_{klkl} and $2(1 - \delta_{k0})$ in Eqs. (11) are added to their solvability conditions. The former (including γ_{klkl}) can contribute to the dynamics whether the modes involved are asymmetric or not. The latter (including $2(1 - \delta_{k0})$) can contribute to the dynamics only when the modes involved are asymmetric ($k \neq 0$). In this study we consider the internal resonance condition including two distinct natural frequencies (ω_{CD} and ω_{NM}) corresponding to asymmetric modes. Then the ratio of two frequencies must be three to one ($\omega_{NM} \approx 3\omega_{CD}$). Furthermore, the terms $R_{kl}^{A,B}$ (representing modal interactions) vanish unless the ratio of the numbers of nodal diameters of two modes is either three to one ($N = 3C$) or one to one ($N = C$). These facts have been clarified by observing Eq. (29) and Table A1 in Yeo and Lee [5]. In view of the three-to-one case investigated by Lee and Yeo [7], it has been found that terms including $2(1 - \delta_{k0})$ vanish even though all modes involved are asymmetric. We observed that in order for the terms not to vanish the ratio must be one to one. This is why we need to investigate the effect of the number of nodal diameters on non-linear interactions in asymmetric vibrations of the plate by examining the one-to-one case ($N = C$).

In order to consider the internal resonance condition $\omega_{CM} \approx 3\omega_{CD}$ and the external resonance condition $\lambda \approx \omega_{CD}$, we introduce detuning parameters, σ_1 and σ_2, as follows:

$$\omega_{CM} = 3\omega_{CD} + \varepsilon\sigma_1, \quad \lambda = \omega_{CD} + \varepsilon\sigma_2. \tag{12, 13}$$

In this case

$$R_{CM}^A = Q_{CM}A_{CD}^2 B_{CD}e^{-i\sigma_1 T_1}, \quad R_{CM}^B = Q_{CM}\bar{A}_{CD}\bar{B}_{CD}^2 e^{i\sigma_1 T_1}, \tag{14a, b}$$

$$R_{CD}^A = \left(Q_{CD}\bar{A}_{CD}\bar{B}_{CD}A_{CM} + Q_{CM}\bar{B}_{CD}^2 B_{CM}\right)e^{i\sigma_1 T_1}, \tag{14c}$$

$$R_{CD}^B = \left(Q_{CD}A_{CD}B_{CD}\bar{B}_{CM} + Q_{CM}A_{CD}^2 \bar{A}_{CM}\right)e^{-i\sigma_1 T_1}, \tag{14d}$$

$$R_{kl}^{A,B} = 0 \quad \text{for } kl \neq CD, \; CM, \tag{14e}$$

$$N_{CD}^A = \tfrac{1}{2}P_{CD}e^{i(\sigma_2 T_1 + \tau_{CD})}, \quad N_{CD}^B = \tfrac{1}{2}P_{CD}e^{-i(\sigma_2 T_1 - \tau_{CD})} \tag{15a, b}$$

$$N_{kl}^{A,B} = 0 \quad \text{for } kl \neq CD, \tag{15c}$$

where the Q_{CM} and Q_{CD} are constants given in Appendix A. Next we let

$$A_{nm} = \tfrac{1}{2}a_{nm}e^{i\alpha_{nm}}, \quad B_{nm} = \tfrac{1}{2}b_{nm}e^{i\beta_{nm}}, \tag{16a, b}$$

where the $a_{nm}, b_{nm}, \alpha_{nm}$ and β_{nm} are real functions of T_1. Substituting Eqs. (14)–(16) into (11) and separating the result into real and imaginary parts, we obtain

$$\begin{aligned}
&\omega_{kl}(a_{kl}' + c_{kl}a_{kl}) - \tfrac{1}{4}(1 - \delta_{k0})b_{kl}\hat{s}_{kl}^s \\
&\quad - \tfrac{1}{8}\delta_{kC}\delta_{lD}\left(Q_{CD}a_{CD}b_{CD}a_{CM}\sin\tilde{\mu}_A + Q_{CM}b_{CD}^2 b_{CM}\sin\tilde{\mu}_B\right) \\
&\quad + \tfrac{1}{8}\delta_{kC}\delta_{lM}Q_{CM}a_{CD}^2 b_{CD}\sin\tilde{\mu}_A - \tfrac{1}{2}\delta_{kC}\delta_{lD}P_{CD}\sin\mu_{CD}^a = 0,
\end{aligned} \tag{17a}$$

$$\begin{aligned}
&\omega_{kl}(b_{kl}' + c_{kl}b_{kl}) + \tfrac{1}{4}(1 - \delta_{k0})a_{kl}\hat{s}_{kl}^s \\
&\quad - \tfrac{1}{8}\delta_{kC}\delta_{lD}(Q_{CD}a_{CD}b_{CD}b_{CM}\sin\tilde{\mu}_B + Q_{CM}a_{CD}^2 a_{CM}\sin\tilde{\mu}_A) \\
&\quad + \tfrac{1}{8}\delta_{kC}\delta_{lM}Q_{CM}a_{CD}b_{CD}^2 \sin\tilde{\mu}_B - \tfrac{1}{2}\delta_{kC}\delta_{lD}P_{CD}\sin\mu_{CD}^b = 0,
\end{aligned} \tag{17b}$$

$$\omega_{kl}a_{kl}\alpha'_{kl} + \tfrac{1}{8}a_{kl}(s_{kl} - \gamma_{klkl}a_{kl}^2) + \tfrac{1}{4}(1 - \delta_{k0})b_{kl}\hat{s}_{kl}^c$$
$$+ \tfrac{1}{8}\delta_{kC}\delta_{lD}\big(Q_{CD}a_{CD}b_{CD}a_{CM}\cos\tilde{\mu}_A + Q_{CM}b_{CD}^2 b_{CM}\cos\tilde{\mu}_B\big)$$
$$+ \tfrac{1}{8}\delta_{kC}\delta_{lM}Q_{CM}a_{CD}^2 b_{CD}\cos\tilde{\mu}_A + \tfrac{1}{2}\delta_{kC}\delta_{lD}P_{CD}\cos\mu_{CD}^a = 0, \tag{17c}$$

$$\omega_{kl}b_{kl}\beta'_{kl} + \tfrac{1}{8}b_{kl}(s_{kl} - \gamma_{klkl}b_{kl}^2) + \tfrac{1}{4}(1 - \delta_{k0})a_{kl}\hat{s}_{kl}^c$$
$$+ \tfrac{1}{8}\delta_{kC}\delta_{lD}\big(Q_{CD}a_{CD}b_{CD}b_{CM}\cos\tilde{\mu}_B + Q_{CM}a_{CD}^2 a_{CM}\cos\tilde{\mu}_A\big)$$
$$+ \tfrac{1}{8}\delta_{kC}\delta_{lM}Q_{CM}a_{CD}b_{CD}^2\cos\tilde{\mu}_B + \tfrac{1}{2}\delta_{kC}\delta_{lD}P_{CD}\cos\mu_{CD}^b = 0, \tag{17d}$$

where primes denote differentiation with respect to T_1,

$$s_{kl} = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\gamma_{klnm}(a_{nm}^2 + b_{nm}^2), \tag{18a}$$

$$\hat{s}_{kl}^s = \sum_{m=1}^{\infty}\hat{\gamma}_{klkm}a_{km}b_{km}\sin(\alpha_{km} - \beta_{km} - \alpha_{kl} + \beta_{kl}), \tag{18b}$$

$$\hat{s}_{kl}^c = \sum_{m=1}^{\infty}(1 - \delta_{ml})\hat{\gamma}_{klkm}a_{km}b_{km}\cos(\alpha_{km} - \beta_{km} - \alpha_{kl} + \beta_{kl}), \tag{18c}$$

$$\mu_{CD}^a = \sigma_2 T_1 + \tau_{CD} - \alpha_{CD}, \quad \mu_{CD}^b = \sigma_2 T_1 - \tau_{CD} - \beta_{CD}, \tag{19a, b}$$

$$\tilde{\mu}_A = \sigma_1 T_1 - 2\alpha_{CD} - \beta_{CD} + \alpha_{CM}, \quad \tilde{\mu}_B = \sigma_1 T_1 - \alpha_{CD} - 2\beta_{CD} + \beta_{CM}. \tag{19c, d}$$

Each equilibrium solution of the system of autonomous ordinary differential equations to be obtained from system (17)–(19) is corresponding to a steady state response. The steady state response to the first order approximation is given as follows:

$$w = w_{CD} + w_{CM} + O(\varepsilon), \tag{20}$$

where

$$w_{CD} = \phi_{CD}\{a_{CD}\cos(\lambda t - \mu_{CD}^a + C\theta + \tau_{CD}) + b_{CD}\cos(\lambda t - \mu_{CD}^b - C\theta - \tau_{CD})\}, \tag{21}$$

$$w_{CM} = \phi_{CM}\{a_{CM}\cos(3\lambda t - 2\mu_{CD}^a - \mu_{CD}^b + \tilde{\mu}_A + C\theta + \tau_{CD})$$
$$+ b_{CM}\cos(3\lambda t - \mu_{CD}^a - 2\mu_{CD}^b + \tilde{\mu}_B - C\theta - \tau_{CD})\}. \tag{22}$$

Each of the w_{CD} and w_{CM} is a superposition of two traveling wave components.

If $a_{CD} = b_{CD}$, $a_{CM} = b_{CM}$, $\mu_{CD}^a = \mu_{CD}^b$ and $\tilde{\mu}_A = \tilde{\mu}_B$, Eqs. (21) and (22) can be reduced as follows:

$$w_{CD} = 2\phi_{CD}a_{CD}\cos(\lambda t - \mu_{CD}^a)\cos(C\theta + \tau_{CD}), \tag{23}$$

$$w_{CM} = 2\phi_{CM}a_{CM}\cos(3\lambda t - 3\mu_{CD}^a + \tilde{\mu}_A)\cos(C\theta + \tau_{CD}). \tag{24}$$

Now each of the w_{CD} and w_{CM} becomes a standing wave component.

3. Numerical example

Pursuing the internal resonance condition $\omega_{CM} \approx 3\omega_{CD}$, we consider the case of $C = 1$ for the convenience. In order to choose a proper value of the stiffness of the elastic foundation, K, we plot the variations of the natural frequencies ω_{1M} and $3\omega_{1D}$ with K in Fig. 2. For a numerical example we choose the case of $K = 1239 \approx K^*$ (intersection of ω_{13} and $3\omega_{11}$ in Fig. 2), which gives natural frequencies $\omega_{11} = 41.7733$ and $\omega_{13} = 125.348$. Then we have an internal resonance condition $\omega_{13} \approx 3\omega_{11}$ with the internal detuning parameter $\varepsilon\sigma_1 = 0.0278555$. Considering a primary resonance $\lambda \approx \omega_{11}$ (the lower mode is directly excited) and substituting the relations of $C = D = 1$ and $M = 3$ into Eqs. (17)–(19), we obtain a simplified system of ordinary differential equations for the non-decaying amplitudes as follows:

$$\omega_{11}(a'_{11} + c_{11}a_{11}) + \tfrac{1}{4}\hat{\gamma}_{1113}b_{11}a_{13}b_{13}\sin(\alpha_{11} - \beta_{11} - \alpha_{13} + \beta_{13})$$
$$- \tfrac{1}{8}\left(Q_{11}a_{11}b_{11}a_{13}\sin\tilde{\mu}_A + Q_{13}b_{11}^2b_{13}\sin\tilde{\mu}_B\right) - \tfrac{1}{2}P_{11}\sin\mu_{11}^a = 0, \tag{25a}$$

$$\omega_{11}(b'_{11} + c_{11}b_{11}) - \tfrac{1}{4}\hat{\gamma}_{1113}a_{11}a_{13}b_{13}\sin(\alpha_{11} - \beta_{11} - \alpha_{13} + \beta_{13})$$
$$- \tfrac{1}{8}\left(Q_{11}a_{11}b_{11}b_{13}\sin\tilde{\mu}_B + Q_{13}a_{11}^2a_{13}\sin\tilde{\mu}_A\right) - \tfrac{1}{2}P_{11}\sin\mu_{11}^b = 0, \tag{25b}$$

$$\omega_{11}a_{11}\alpha'_{11} + \tfrac{1}{8}a_{11}\left\{\gamma_{1111}(a_{11}^2 + 2b_{11}^2) + 2\gamma_{1113}(a_{13}^2 + b_{13}^2)\right\}$$
$$+ \tfrac{1}{4}\hat{\gamma}_{1113}b_{11}a_{13}b_{13}\cos(\alpha_{11} - \beta_{11} - \alpha_{13} + \beta_{13})$$
$$+ \tfrac{1}{8}\left(Q_{11}a_{11}b_{11}a_{13}\cos\tilde{\mu}_A + Q_{13}b_{11}^2b_{13}\cos\tilde{\mu}_B\right) + \tfrac{1}{2}P_{11}\cos\mu_{11}^a = 0, \tag{25c}$$

$$\omega_{11}b_{11}\beta'_{11} + \tfrac{1}{8}b_{11}\left\{\gamma_{1111}(2a_{11}^2 + b_{11}^2) + 2\gamma_{1113}(a_{13}^2 + b_{13}^2)\right\}$$
$$+ \tfrac{1}{4}\hat{\gamma}_{1113}a_{11}a_{13}b_{13}\cos(\alpha_{11} - \beta_{11} - \alpha_{13} + \beta_{13})$$
$$+ \tfrac{1}{8}\left(Q_{11}a_{11}b_{11}b_{13}\cos\tilde{\mu}_B + Q_{13}a_{11}^2a_{13}\cos\tilde{\mu}_A\right) + \tfrac{1}{2}P_{11}\cos\mu_{11}^b = 0, \tag{25d}$$

$$\omega_{13}(a'_{13} + c_{13}a_{13}) - \tfrac{1}{4}\hat{\gamma}_{1311}a_{11}b_{11}b_{13}\sin(\alpha_{11} - \beta_{11} - \alpha_{13} + \beta_{13}) + \tfrac{1}{8}Q_{13}a_{11}^2b_{11}\sin\tilde{\mu}_A = 0, \tag{26a}$$

$$\omega_{13}(b'_{13} + c_{13}b_{13}) + \tfrac{1}{4}\hat{\gamma}_{1311}a_{11}b_{11}a_{13}\sin(\alpha_{11} - \beta_{11} - \alpha_{13} + \beta_{13}) + \tfrac{1}{8}Q_{13}a_{11}b_{11}^2\sin\tilde{\mu}_B = 0, \tag{26b}$$
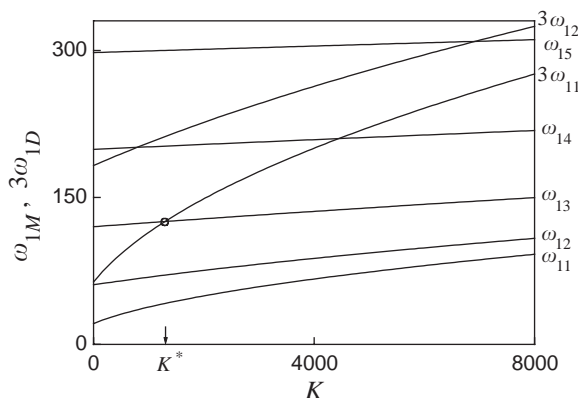

Fig. 2. Variations of the natural frequencies ω_{1M} and $3\omega_{1D}$ with the stiffness of the foundation, K.

$$\omega_{13}a_{13}\alpha'_{13} + \tfrac{1}{8}a_{13}\{2\gamma_{1311}(a_{11}^2 + b_{11}^2) + \gamma_{1313}(a_{13}^2 + 2b_{13}^2)\}$$
$$+ \tfrac{1}{4}\hat{\gamma}_{1311}a_{11}b_{11}b_{13}\cos(\alpha_{11} - \beta_{11} - \alpha_{13} + \beta_{13}) + \tfrac{1}{8}Q_{13}a_{11}^2 b_{11}\cos\tilde{\mu}_A = 0, \qquad (26c)$$

$$\omega_{13}b_{13}\beta'_{13} + \tfrac{1}{8}b_{13}\{2\gamma_{1311}(a_{11}^2 + b_{11}^2) + \gamma_{1313}(2a_{13}^2 + b_{11}^2)\}$$
$$+ \tfrac{1}{4}\hat{\gamma}_{1311}a_{11}b_{11}a_{13}\cos(\alpha_{11} - \beta_{11} - \alpha_{13} + \beta_{13}) + \tfrac{1}{8}Q_{13}a_{11}b_{11}^2\cos\tilde{\mu}_B = 0, \qquad (26d)$$

$$\mu_{11}^a = \sigma_2 T_1 + \tau_{11} - \alpha_{11}, \quad \mu_{11}^b = \sigma_2 T_1 - \tau_{11} - \beta_{11}, \qquad (27a, b)$$

$$\tilde{\mu}_A = \sigma_1 T_1 - 2\alpha_{11} - \beta_{11} + \alpha_{13}, \quad \tilde{\mu}_B = \sigma_1 T_1 - \alpha_{11} - 2\beta_{11} + \beta_{13}. \qquad (27c, d)$$

Each equilibrium solution of a system of autonomous ordinary differential equations to be obtained from systems (26) and (27) is corresponding to a steady state response ($a'_{11} = b'_{11} = a'_{13} = b'_{13} = \mu_{11}^{a'} = \mu_{11}^{b'} = \tilde{\mu}'_A = \tilde{\mu}'_B = 0$).

In Fig. 3 the amplitudes a_{11}, b_{11}, a_{13} and b_{13} are plotted as functions of the external detuning parameter $\varepsilon\sigma_2 = \hat{\sigma}_2$ when $\{v, \varepsilon, \varepsilon c, \varepsilon P_{11}, \tau_{11}\} = \{1/3, 0.001, 0.001, 5.0, 0.0\}$. Solid and dotted lines denote, respectively, stable and unstable responses. The abbreviations SS, US, ST and UT denote, respectively, stable standing, unstable standing, stable traveling and unstable traveling wave components.
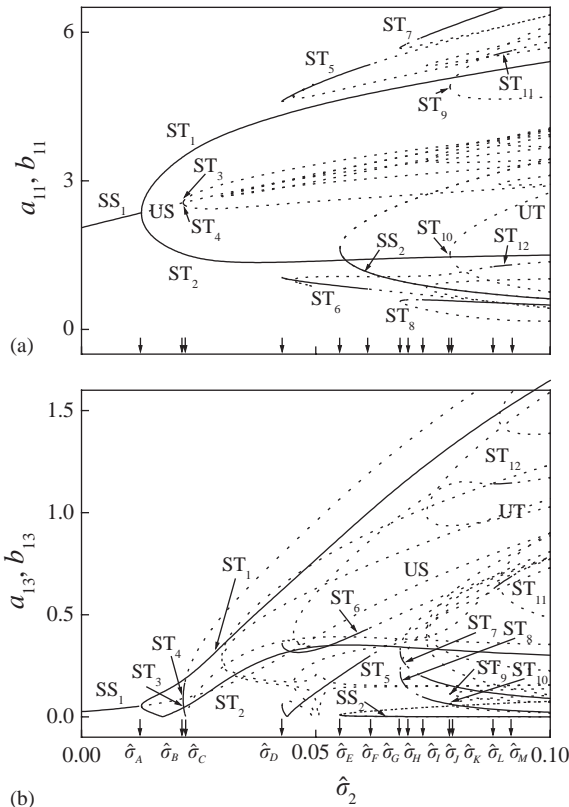

Fig. 3. Variations of the amplitudes with detuning parameter $\hat{\sigma}_2$ when $\varepsilon P_{11} = 5$. ——, stable; - - - -, unstable.

The response curves are shown to have two pitchfork, ten saddle-node and seven Hopf bifurcation points. At one of two pitchfork bifurcation points, $\hat{\sigma}_A(0.012797)$, the stable standing wave component SS_1 bifurcates into two stable traveling wave components ST_1 and ST_2, and one unstable standing wave component US. At the other pitchfork bifurcation point, $\hat{\sigma}_B(0.021699)$, the unstable standing wave component US appears to bifurcate into two stable traveling wave components ST_3 and ST_4, and one unstable standing wave component US. In fact, at $\hat{\sigma}_2 = 0.021675$ not identified in the figure, the unstable component US becomes a stable component, which bifurcates into ST_3 and ST_4. At four saddle-node bifurcation points out of 10, $\hat{\sigma}_D$ (0.042807), $\hat{\sigma}_E$ (0.055153), $\hat{\sigma}_G$ (0.067977) and $\hat{\sigma}_J$ (0.078646), one stable response and one unstable response generate. At the other saddle-node bifurcation points (not marked in the figure), two unstable responses generate.

It is observed that only traveling wave components experience Hopf bifurcations. The components change their stability at seven Hopf bifurcation points $\hat{\sigma}_C$ (0.022026), $\hat{\sigma}_F$ (0.060973), $\hat{\sigma}_H$ (0.067977), $\hat{\sigma}_I$ (0.072787), $\hat{\sigma}_K$ (0.078647), $\hat{\sigma}_L$ (0.088036) and $\hat{\sigma}_M$ (0.091731). For instance, when $\hat{\sigma}_J < \hat{\sigma}_2 < \hat{\sigma}_K$ and $\hat{\sigma}_L < \hat{\sigma}_2 < \hat{\sigma}_M$, there exist seven stable steady state responses, which consist of six traveling wave components and one standing wave component. When $\hat{\sigma}_E < \hat{\sigma}_2 < \hat{\sigma}_F$, $\hat{\sigma}_G < \hat{\sigma}_2 < \hat{\sigma}_H$, $\hat{\sigma}_I < \hat{\sigma}_2 < \hat{\sigma}_J$, $\hat{\sigma}_K < \hat{\sigma}_2 < \hat{\sigma}_L$ and $\hat{\sigma}_2 < \hat{\sigma}_M$, there exist five stable steady state responses, which consist of four traveling wave components and one standing wave component. When $\hat{\sigma}_B < \hat{\sigma}_2 < \hat{\sigma}_C$ and $\hat{\sigma}_D < \hat{\sigma}_2 < \hat{\sigma}_E$, there exist four stable steady state responses, which are traveling wave components. When $\hat{\sigma}_F < \hat{\sigma}_2 < \hat{\sigma}_G$ and $\hat{\sigma}_H < \hat{\sigma}_2 < \hat{\sigma}_I$, there exist three stable steady state responses, which consist of two traveling wave components and one standing wave component. When $\hat{\sigma}_F < \hat{\sigma}_2$, $\hat{\sigma}_H < \hat{\sigma}_2 < \hat{\sigma}_I$ and $\hat{\sigma}_K < \hat{\sigma}_2$, two pairs of traveling wave components, $\{ST_5, ST_6\}$, $\{ST_7, ST_8\}$ and $\{ST_9, ST_{10}\}$, respectively, become unstable. Another pair of stable components $\{ST_{11}, ST_{12}\}$ loses stability at $\hat{\sigma}_2 = \hat{\sigma}_L$ and $\hat{\sigma}_M$. These instabilities imply that there may exist quasi-periodic or chaotic responses generated from these pairs. Exploring this type of response, however, is beyond the scope of this work.

Conclusively, non-vanishing amplitudes of indirectly excited modes (a_{13} and b_{13}) tell us modal interactions between lower (ω_{11}) and higher (ω_{13}) modes. The characteristics of the responses in this work are much more complicated than the three-to-one case ($N = 3C$) [7].

4. Conclusions

In order to investigate the effect of the number of nodal diameters on non-linear interactions in asymmetric vibrations of a circular plate on an elastic foundation, a primary resonance of the plate is considered. The plate is assumed to have an internal resonance in which the ratio of the natural frequencies of two asymmetric modes is three to one. It has been found that in order for the modes to interact, the ratio of the numbers of nodal diameters of two modes must be either three to one or one to one.

In this study the one-to-one case, in which the modes have the same number of nodal diameters, is examined. More precisely, we consider a primary resonance case with an internal resonance condition $\omega_{13} \approx 3\omega_{11}$ and an external resonance condition $\lambda \approx \omega_{11}$ (the lower mode is directly excited). Non-vanishing amplitudes of indirectly excited modes (a_{13} and b_{13}) tell us modal interactions between lower (ω_{11}) and higher (ω_{13}) modes. The characteristics of the interactions are much more complicated than the three-to-one case ($N = 3C$ or $\omega_{32} \approx 3\omega_{11}$) studied previously [7].

Acknowledgements

Appendix A

1. Eq. (7):
The linear symmetric vibration modes $\phi_{nm}(r)$ corresponding to the natural frequencies ω_{nm} are given by

$$\phi_{nm} = \kappa_{nm}\left[\mathrm{J}_n(\eta_{nm}r) - \frac{\mathrm{J}_n(\eta_{nm})}{\mathrm{I}_n(\eta_{nm})}\,\mathrm{I}_n(\eta_{nm}r)\right]. \tag{A.1}$$

The κ_{nm} are chosen so that

$$\int_0^1 r\phi_{nm}^2\,\mathrm{d}r = 1. \tag{A.2}$$

The function J_n are Bessel function of the first kind of order n and the function I_n are modified Bessel function of the first kind of order n. the η_{nm} are the roots of $\mathrm{I}_n(\eta)\mathrm{J}_n'(\eta) - \mathrm{I}_n'(\eta)\mathrm{J}_n(\eta) = 0$. The natural frequencies ω_{nm} are related with the eigenvalues η_{nm} by the equation $\omega_{nm}^2 = \eta_{nm}^4 + K$. $\phi_{-nm} = \phi_{nm}$, $\omega_{-nm} = \omega_{nm}$ and $A_{-nm} = B_{nm}$.

2. Eq. (11):

$$\gamma_{klnm} = \Gamma(kl, kl, nm, -nm) + \Gamma(kl, -nm, kl, nm) + \Gamma(kl, nm, -nm, kl), \tag{A.3}$$

$$\hat{\gamma}_{klkm} = \Gamma(kl, km, km, -kl) + \Gamma(kl, -kl, km, km) + \Gamma(kl, km, -kl, km), \tag{A.4}$$

where

$$\Gamma(kl, cd, nm, pq) = \sum_{b=1}^{\infty} G(nm, pq; ab)\hat{G}(cd, ab; kl), \quad a = k - c, \quad p = k - c - n, \tag{A.5}$$

$$G(nm, pq; ab) = \xi_{ab}^{-4}\int_0^1 r\psi_{ab}E(nm, pq)\,\mathrm{d}r, \tag{A.6}$$

$$\hat{G}(cd, ab; kl) = \int_0^1 r\phi_{kl}\widehat{E}(cd, ab)\,\mathrm{d}r, \tag{A.7}$$

$$E(nm, pq) = \frac{-np}{r^2}\left(\phi_{nm}' - \frac{\phi_{nm}}{r}\right)\left(\phi_{pq}' - \frac{\phi_{pq}}{r}\right) - \frac{1}{2r}(\phi_{nm}'\phi_{pq}')' + \frac{1}{2r^2}(p^2\phi_{nm}''\phi_{pq} + n^2\phi_{pq}''\phi_{nm}), \tag{A.8}$$

$$\widehat{E}(cd, ab) = \frac{\phi_{cd}''}{r}\left(\psi_{ab}' - \frac{a^2}{r}\psi_{ab}\right) + \frac{\psi_{ab}''}{r}\left(\phi_{cd}' - \frac{c^2}{r}\phi_{cd}\right) + \frac{2ac}{r^2}\left(\psi_{ab}' - \frac{1}{r}\psi_{ab}\right)\left(\phi_{cd}' - \frac{1}{r}\phi_{cd}\right), \tag{A.9}$$

and

$$\psi_{ab} = \tilde{\kappa}_{ab}[\mathrm{J}_a(\xi_{ab}r) - \tilde{c}_{ab}\mathrm{I}_a(\xi_{ab}r)]. \tag{A.10}$$

The $\tilde{\kappa}_{ab}$ are chosen so that

$$\int_0^1 r\psi_{ab}^2\,\mathrm{d}r = 1, \tag{A.11}$$

$$\tilde{c}_{ab} = \frac{[a(a+1)(v+1) - \xi_{ab}^2]\mathrm{J}_a(\xi_{ab}) - \xi_{ab}(v+1)\mathrm{J}_{a-1}(\xi_{ab})}{[a(a+1)(v+1) + \xi_{ab}^2]\mathrm{I}_a(\xi_{ab}) - \xi_{ab}(v+1)\mathrm{I}_{a-1}(\xi_{ab})} \tag{A.12}$$

and the ξ_{ab} are the roots of

$$a^2(a+1)(v+1)[\mathrm{J}_a(\xi_{ab}) - \tilde{c}_{ab}\mathrm{I}_a(\xi_{ab})] - a^2\xi_{ab}(v+1)[\mathrm{J}_{a-1}(\xi_{ab}) - \tilde{c}_{ab}\mathrm{I}_{a-1}(\xi_{ab})]$$
$$+ a\xi_{ab}^2[\mathrm{J}_a(\xi_{ab}) + \tilde{c}_{ab}\mathrm{I}_a(\xi_{ab})] - \xi_{ab}^3[\mathrm{J}_{a-1}(\xi_{ab}) + \tilde{c}_{ab}\mathrm{I}_{a-1}(\xi_{ab})] = 0. \tag{A.13}$$

3. Eq. (14):

$$Q_{CM} = 2\Gamma(CM, CD, CD, -CD) + \Gamma(CM, -CD, CD, CD), \tag{A.14}$$

$$Q_{CD} = 2\{\Gamma(CD, -CD, CD, CM) + \Gamma(CD, CD, CM, -CD) + \Gamma(CD, CM, CD, -CD)\}. \tag{A.15}$$

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